I) What we need to define $\mathcal{F}(M)$ as an ungraded A_∞-cat. over $\mathbb{Z}/2$:

Partial definition: the Fukaya cat. $\mathcal{F}(M)$ is the A_∞-cat. with

- objects = { Lagrangian submflds of M } (exact, closed, ...)
- $\hom(L_0, L_1)$ = vector space gen'd by $L_0 \cap L_1$ (assuming $L_0 \pitchfork L_1$).
- $\mu^d : \hom(L_{d-1}, L_d) \otimes \ldots \otimes \hom(L_0, L_1) \to \hom(L_0, L_d)$
  given by counts of $(d+1)$-pointed J-holom. discs.

Problems:
- what if L_0, L_1 not transverse? (eg. if $L_0 = L_1$?)
- consistency: transversality of moduli spaces?
  gluing consistency as discs degenerate?
  (so A_∞-relations still hold).

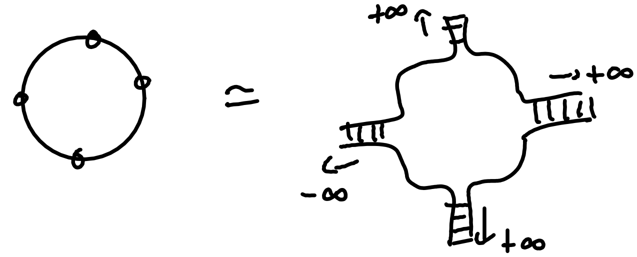
Def: Given $L_0, L_1 \subset M$, a Floer datum := (H, J) where
H = t-dependent Hamiltonian $M \to \mathbb{R}$
J = t-dependent almost-cx. structure
($t \in [0,1]$)
st. $L_1 \pitchfork \phi_H(L_0)$ transversely. (ϕ_H = time 1 flow of H)
and st. J is regular for H-perturbed holom. strips, ie. moduli spaces of solutions of

$$\bar{\partial} u := \frac{\partial u}{\partial s} + J(t)\left(\frac{\partial u}{\partial t} - X_H(t)\right) = 0 \qquad \left(u : Z = \underset{s}{\mathbb{R}} \times \underset{t}{[0,1]} \to M\right)$$

are smooth mflds of the expected dimension.

★ We'll fix a Floer datum for every pair of Lagrangians.

Def: If S is a $(d+1)$-pointed disc, a set of strip-like ends for S is a collection $\{\varepsilon_\zeta\}$, ζ boundary marked point, of holom. maps $\varepsilon_\zeta : \underset{\mathbb{R}_+ \times [0,1]}{Z^+} \to S$
with $\mathbb{R}^+ \times \{0,1\} \to \partial S$ for the marked pts $1, \ldots, d$ ("positive" punctures = inputs)
and $\varepsilon_\zeta : Z^- = \mathbb{R}_- \times [0,1] \to S$ similarly for marked pt 0 ("neg." puncture = output).

ie: choose charts near punctures on 

If S is thought of as punctured at ζ's, a set of <u>Lagrangian labels</u> for S is an assignment of a Lagr. submfld L_i to each component C_i of ∂S.

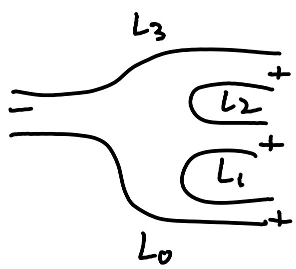
<u>Note</u>: we then have a chosen Floer datum at each end, by the above.

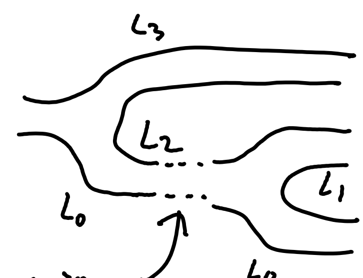
* Given a pointed disk S with strip-like ends & Lagr. labels, define a <u>perturbation datum</u> (K, J) to consist of:

$\begin{cases} \bullet \text{ a 1-form } K \in \Omega^1(S, \mathcal{H}) \\ \qquad\qquad\qquad \hookrightarrow \text{space of Hamiltonians} \\ \bullet \text{ a domain-dep}^{t} \text{ acs. } J \in C^\infty(S, \mathcal{J}). \end{cases}$

st. $\begin{cases} \bullet \text{ in each strip-like end, } (K, J)_{|\varepsilon_\zeta} \equiv (H\,dt, J) \text{ the given Floer data for pair } (L_i, L_j). \\ \bullet\ K(\xi)_{|L_i} = 0 \text{ for } \xi \in TC_i \text{ a vector tangent to } \partial\text{-component } C_i \text{ labelled by } L_i. \end{cases}$

<u>Issue</u>: need to choose these perturbation data <u>consistently</u>, ie. as S <u>degenerates</u> to a pair of disks, need perturbation in the newly formed strip like end to agree with previously chosen Floer data for the appropriate Lagrangians.

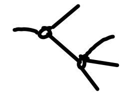
ie. if 

degenerate to 

then in this region (K,J) needs to agree with Floer data for (L_0, L_2).

Do this inductively on associahedra:

$R^4 = \longmapsto$, smoothing near an end = choose chart $g: \overset{\text{pt}}{R^3} \times \overset{\text{pt}}{R^3} \times \overset{\text{gluing param.}}{[0,1)} \to R^4$

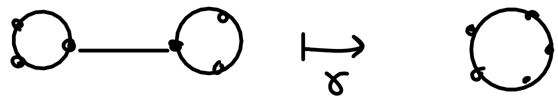
then $\mathcal{R}^4 =$  here, 

choose gluing chart

$\gamma : \mathcal{R}^3 \times \mathcal{R}^4 \times [0,1) \to \mathcal{R}^5 \dots$

pt

s.t. 1) strip-like ends agree across gluing:



the strip-like ends of glued disc agree with those of components (for small values of gluing parameter)

2) similarly for perturbation data

(however, only require pert. data to strictly agree on "thin part" i.e. strip like ends + glued pieces, and to converge to pert. data of pieces as gluing parameter $\to 0$ on thick parts).

II) gradings: we know (assuming transversality) that for fixed domain

$\dim \mathcal{M} = \operatorname{ind} D_{\bar\partial}$ = some topological quantity

↳ Maslov index.

To get a $\mathbb{Z}$-grading on the Fukaya cat., need to assume: $2c_1(M) = 0$.

& will only consider graded Lagrangian submflds.

Since $2c_1(M) = 0$, $\exists$ global trivialization of $(\Lambda^n_{\mathbb{C}} TM)^{\otimes 2}$, i.e. a nonvanishing section Ω of it, and we can consider the "phase-squared" map $\phi(L) = (\mathrm{vol}_L)^2/\Omega : L \to S^1$

In fact, ϕ is a map defined on the Lagrangian grassmannian bundle

$$\begin{array}{ccc} LGr(M) & \xrightarrow[\phi]{} & S^1 \\ \downarrow & & \\ M & & \end{array}$$

(Note: in $\mathbb{C}^n$, Lag. planes = image of $A : \mathbb{R}^n \to \mathbb{C}^n$, $A \in U(n)/O(n)$, then ϕ is essentially $(\det A)^2$)

Rmk: $\phi : LGr_n \to S^1$ generates $H^1(LGr_n, \mathbb{Z})$.

This implies that (under our assumption of $2c_1(M) = 0$) there is a bundle

$$\begin{array}{c} \widetilde{LGr}(M) \\ \downarrow \\ LGr(M) \\ \downarrow \\ M \end{array}$$

fiberwise universal cover.

Def: The Maslov class of L is $\mu \in H^1(L, \mathbb{Z})$ def'd by phase $[\phi(L)] \in [L, S^1] \simeq H^1(L, \mathbb{Z})$.

If $\mu = 0$ then L lifts to a section of $\widetilde{LGr}(M)$. (and conversely)

$$\begin{tikzcd} & \widetilde{LGr}(M) \arrow[d] \\ & LGr(M) \arrow[d] \\ L \arrow[uur, dashed, "?"] \arrow[ur, "TL"] \arrow[r, "\text{incl.}"] & M \end{tikzcd}$$

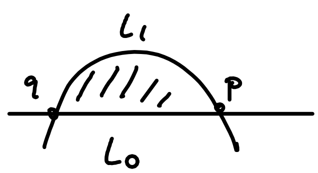
Def: A graded Lagrangian $\tilde{L}$ is a pair (L, ℓ) where $L \subset M$ Lagrangian (necess. with $\mu = 0$) and ℓ is a lift to $\widetilde{LGr(M)}$

Def: If $\tilde{L}_0, \tilde{L}_1$ graded Lagrangians, and $x \in L_0 \cap L_1$, transverse int., then $\exists!\ \gamma : [0,1] \to \widetilde{LGr}$ connecting the lifts of $\widetilde{T_x L_0}$ and $\widetilde{T_x L_1}$ up to homotopy

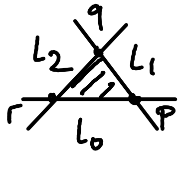
The index $i_{(\tilde{L}_0, \tilde{L}_1)}(x)$ is related to winding number of $\phi(\gamma)$ (see next time). – in dim 1 it's $\lceil \text{winding} \rceil$

Index formula: $$\mathrm{Ind}(D_\zeta) = i(x_0) - \sum_{\zeta \neq \zeta_0} i(x_\zeta).$$

Ex:



$\mathrm{Ind} = i(q) - i(p) = 1$.



eg $\phi(L_0) = 0$, $0 < \phi(L_1) < 1$, $1 < \phi(L_2) < 2$ $\Rightarrow$ $i(p) = i(q) = 1$, $i(r) = 2$

$\mathrm{Ind} = 2 - (1+1) = 0$.